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Introduction to E. Nelson's internal set theory*

Abstract. An axiomatic approach to Non-standard Analysis by E. Nelson is presented in a simplified form. The main aim of the article is strictly the popularization of NSA, and not its foundations. No special preparation in mathematical logic is required from the reader but it is assumed that he (she) is familiar with elementary calculus and linear algebra.

Introduction

Non-standard Analysis is an enlargement of Standard or Ordinary Mathematics. It provides a strictly logical foundation of Leibniz's infinitesimals. It enables us to better understand and simplify a lot of Standard Mathematics. It also advances new mathematical problems.

In Non-standard Analysis (NSA in short), the old problem of the substantiation of differential and integral calculus with the application of infinitesimals was solved. This problem seemed to be unsolvable in the times of G. Leibniz and L. Euler. NSA has changed the face of the whole of Mathematics: it is a new mathematical outlook. It is necessary to emphasise that NSA does not object or contradict the Ordinary Mathematics (OM in short). NSA extends and supplements OM. This means that all objects which exist in OM also exist in NSA, and all statements which are true in OM remain true in NSA. NSA often simplifies OM and makes it more transparent. NSA states new mathematical theorems and problems.

NSA was created mainly by A. Robinson (1960), who developed H. Hahn's, T. Skolem's, A. Malcev's, E. Hewitt's, and J. Łoś's ideas. Afterwards, W.A.J. Luxemburg proposed an ultrafilter approach. In this article, we present an axiomatic approach to NSA, due to E. Nelson (Nelson, 1977), which is less difficult to learn and apply. Our exposition, contrary to that of Nelson, is not always

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strictly logical. Our aim is strictly the popularization of NSA, and not its foundations. Other axiomatic approaches were presented in the seventies by K. Hrbaček (Hrbaček, 1979, 1987) and T. Kawai (Kawai, 1981). In our opinion, Nelson's approach is the best for many applications.

Interested readers may find a continuation of this presentation in the same spirit in (Lyantse, Kudryk, 1997). There we provided a more complete picture of NSA. Good supplements to the book are (Albeverio, Fenstad, Høegh-Krohn, Lindstrøm, 1986, Cutland, 1983, Cutland, 1988, Davis, 1977, Diener, 1983, Diener, Reeb, 1989, Goldblatt, 1998, Lutz, Goze, 1981). In the references, we also listed other books which influenced our thinking. Futhermore, we would recommend looking at more recent publications such as (Hrbaček, Lessman, O'Donovan, 2015) and (Kanovei, Reeken, 2004). The latter is the most comprehensive source on various modern non-standard axiomatic theories, including IST.

Some exercises are left to the reader (through courtesy of the author).

1. Sets

Let us begin with a remark that all what is determined and investigated in Mathematics can be considered as a set. (We are not going to pursue a class path here.) For instance, the number zero may be identified with the empty set $\emptyset$, 1 can be defined as the one–element set $\{0\}$. Whenever a natural number n is defined as a set, we can consider $n+1$ as $n \cup \{n\} = \{0, 1, \ldots, n\}$. We consider each function and relation as a set by identifying them with their graphs, and so on.

2. Standard, internal, and external sets

The base of OM (= ordinary mathematics = standard mathematics) is ZFC (= Zermelo–Fraenkel Set Theory with the Axiom of Choice); see e.g. (Luxemburg, Robinson, 1972) or (Robert, 1988).

Sets which are uniquely determined in ZFC are said to be **standard**.

Thus $0, 1, 2, \ldots, 10^{99}, \ldots, 1/2, \sqrt{2}, \pi, \ldots, +, <, \ldots, \sin, \ldots, \mathbb{N}, \mathbb{R}, \mathbb{C}, \ldots, C[0,1], \ldots, L_2(\mathbb{R}), \ldots$ are all standard.

In NSA, elements of standard sets are said to be **internal**.

Thus, in NSA, if $A \in B$ and B is standard, then A can be nonstandard, but it must be internal. This means that for such A (*for arbitrary internal* A) *all laws of* OM *hold.* But due to NSA's methods of the construction of subsets (parts) of a standard set, some statements of OM are not true for these subsets.

If $A \subseteq B$ and B is standard, then A is said to be **external**.

If there is a statement which is true for all standard sets but not true for A, then A is said to be **strictly external**. See below for examples.

Remark 1

If an A is an element of a set B which is standard, internal, or external, then it must be internal. The cause is that each external set is a part of a standard one, and elements of standard sets are internal (which is sometimes repeated for pedagogical reasons). Furthermore, in order for IST to be a closed theory, we ought to assume that the elements of internal sets are internal.

Denote by $\mathbf{S}$, $\mathbf{I}$, $\mathbf{V}$ universes (i.e. totalities) of standard, internal, and external sets respectively. We have

$$\mathbf{S} \subset \mathbf{I} \subset \mathbf{V}.$$

These universes are not sets, but proper classes. We shall see later that $\mathbf{S} \setminus \mathbf{I}$ and $\mathbf{V} \setminus \mathbf{I}$ are not empty.

Remark 2
In this article, we accept the following agreement: "A set" always means "an internal set". If X is a set, then 2^X is the (internal) set of all internal parts of X. If X, Y are sets, then Y^X denotes the (internal) set of all functions f with $\operatorname{dom} f = X$ and $\operatorname{ran} f \subset Y$. Exceptional cases are possible in which it is clear from context that a set under consideration is standard or external.

3. Might of the Word; the word "standard"

The Bible asserts that the Word has the ability to create (see *John 1, 1-4*). Mathematics permanently demonstrates this capacity of words. For instance, take the preposition "*between.*" Let a be a straight line, and A, B its different points. Due to the word "between" we can create the segment $AB := \{X \in a\colon\ X$ is between A and $B\}$. Next, we can define the straight ray $r_{AB} := \{X \in a\colon\ A$ is not between X and $B\}$ and convert r_{AB} into an ordered set by $(\forall\, X, Y \in r_{AB})\,(X < Y \iff X$ is between A and $Y\}$, and so on.

We can also notice that the same word plays a fundamental role in J. Conway's theory of surreal numbers (see Conway, 1976). Unfortunately,the absolute arithmetical continuum created by Conway is unfitting for the needs of the analysis for the time being.

Exercise 1
Let a be a straight line of the plane Σ. Define with the help of "between" two halfplanes Σ_a^1 and Σ_a^2, bounded by a. Hint: Use the concept of a segment which is descended from "between".

Without exaggerration, one can say that the entire OM is created by two words: "*a set*" and "*to belong*". If we wish to extend the OM, we ought to invent a new word. To this end, Robinson has invented the adjective "standard," which came into general use. It is very important to keep in mind that *OM does not know what is "standard."* Therefore, *the laws of OM do not control the property "to be standard"* and the properties which are derived from it.

Let us consider some simple applications of the notion "standard".

4. Classification of reals

It is convenient and comfortable to agree the following. The formula "$st(x)$" denotes "$x \in \mathbf{S}$", i.e. "x is standard"; "$\forall^{st} x\, p(x)$" denotes "$(\forall x)\,(st(x) \Rightarrow p(x)$", i.e. "for any standard x $p(x)$"; "$\exists^{st} x\ p(x)$" denotes "$(\exists x)\,(st(x) \wedge p(x))$", i.e. "for a standard x $p(x)$".

Let $x \in \mathbb{R}$, then "$x \approx 0$" denotes "$(\forall^{st} n \in \mathbb{N})\,(|x| < 1/n)$", i.e. "$x$ is **infinitesimal**". For x, $y \in \mathbb{R}$ "$x \approx y$" means "$x - y \approx 0$", i.e. "x is near y". We write "$x \approx \infty$" for "$(\forall^{st} n \in \mathbb{N})\,(|x| > n)$", and "$x \approx +\infty$" or "$x \approx -\infty$" if, respectively, "$x \approx \infty \wedge x > 0$," or "$x \approx \infty \wedge x < 0$." If $x \approx \infty$, we say that x is **infinitely large**. At last, notation "$x \gg 0$" and "$|x| \ll \infty$" are equivalent to "$x > 0 \wedge x \not\approx 0$" and "$x \not\approx \infty$" respectively. If $|x| \ll \infty$, we say that x is **a limited** number, and if $0 \ll |x| \ll \infty$, x is an **appreciable** number.

Remark 3

It can be shown that the continuity of a standard function f at a standard point a is strictly equivalent to the implication "$x \approx a \implies f(x) \approx f(a)$". This is an example which demonstrates how to use our new concepts.

5. Existence of infinitesimals; the special idealization principle (I_0)

To ensure the existence of infinitesimals, we accept the following axiom:

(I_0) *A standard set is infinite if (and only if) it contains a non-standard element.*

"Non-standard" means "is not standard". We write "$\neg st(x)$" for "$x \notin \mathbf{S}$". It can be shown that (I_0) is a special case of the general principle of idealization (I).

Corollary 1

There exist non-standard natural, rational, real numbers.

Indeed, the sets $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ are standard and infinite. The reader can point out many other examples, for instance: there exist non-standard vectors, transformations, groups, spaces, and so on.

It is useful to formulate (I_0) in a more exact way. As it is known, a set E is said to be *finite* if there exist a number $n \in \mathbb{N}$ and a bijective (that is one-to-one) transformation f which sends E onto $\{1, 2, \ldots, n\}$. In this case, we write $n = \operatorname{card} E$, and n is said to be the *cardinality* or the *quantity of elements* of E. We also write "$\operatorname{fin} E$" for "E is finite." The number $n = \operatorname{card} E$ (which is independent of the choice of f) can be non-standard, but if we write $\operatorname{card} E = n$, we assume that the function f, which is a bijection $E \to \{1, 2, \ldots, n\}$, is *internal*: $f \in \mathbf{I}$ (i.e. its graph is an internal set). We introduce the unary predicate "fin" as follows:

$$\text{“}\operatorname{fin}(x)\text{”} \equiv \text{“}x \text{ is finite.”}$$

The special principle of idealization is as follows:

$$(\forall\, x \in \mathbf{S})\ (\neg \operatorname{fin}(x) \iff (\exists\, y \in x) \neg st(y)), \tag{I_0}$$

where $\neg a$ means the negation of a.

Apply to (I_0) the following laws of Logic: $\neg\neg a \equiv a$, $a \Leftrightarrow b \equiv \neg a \Leftrightarrow \neg b$, $\neg(\exists x)\, p(x) \equiv (\forall\, x)\, \neg p(x)$. We get

$$(\forall\, x \in \mathbf{S})\ (\operatorname{fin}(x) \iff (\forall\, y \in x)\ st(y)). \tag{I_1}$$

This is the *second form* of the special principle of idealization: *a standard set is finite if and only if all its elements are standard.*

Proposition 1
A natural number is infinite if and only of it is non-standard.

Proof. Let $n \in \mathbb{N}$ be standard. Then $\mathbb{N}_n := \{1, \dots, n\}$ is a well-defined set of OM. Therefore, it is standard. Since $\mathbb{N}_n$ is finite, namely $\operatorname{card} \mathbb{N}_n = n \in \mathbb{N}$, each of its elements is standard (see (I_1)). But $(\forall\, p \in \mathbb{N})\ (p \leq n \iff p \in \mathbb{N}_n)$. Since

$$(\forall\, n, p \in \mathbb{N})\ (st(n) \wedge p < n \implies st(p)),$$

we have

$$(\forall\, p \in \mathbb{N})\ (\neg st(p) \iff (\forall^{st} n \in \mathbb{N})\ (p > n)),$$

and we see that

$$(\forall\, p \in \mathbb{N})\ (\neg st(p) \iff p \approx \infty).$$

Remark 4
For $x \in \mathbb{R}$, the formula $|x| \ll \infty$ does not imply that x is standard. For example, the segment $[0, 1]$ is standard and infinite, therefore (see (I_0)) it contains some non-standard numbers.

Exercise 2
Prove that any $n \in \mathbb{Z}$ is limited if and only if it is standard.

Proposition 2
There exist infinitesimal real numbers different from 0.

Proof. Let $\omega \in \mathbb{N}$ and ω be non-standard. Then (see (I_0)) $\omega \approx \infty$, , i.e. $(\forall^{st} n \in \mathbb{N})\, (\omega > n)$. Therefore $(\forall^{st} n \in \mathbb{N})\ (1/\omega < 1/n)$, i.e. $1/\omega \approx 0$. ►

6. The first examples of strictly external sets

In OM, we often use the following subset construction. Let A be a set, and $p(\cdot)$ a property (predicate). Then $B := \{x \in A :\ p(x)\}$ is the set of all x from A which have the property $p(\cdot)$. In NSA, we also apply this construction. For instance, we set

$${}^{st}A := \{x \in A :\ st(x)\}.$$

Thus ${}^{st}A$ denotes the totality of the standard elements of A. According to section 2, if A is standard or internal, ${}^{st}A$ is external as a part of A.

Recall that an external set is said to be *strictly* external if it is non-internal. In other words, it is not subordinate to the laws of OM.

Proposition 3
The set $\mathbb{N} \setminus {}^{st}\mathbb{N} := \{n \in \mathbb{N} :\ \neg st(n)\}$ *is strictly external.*

Proof. In OM, the *least number principle* is known: each non–empty set of natural numbers contains the least number. We have $\mathbb{N} \setminus {}^{st}\mathbb{N} \neq \emptyset$. Set $n := \min(\mathbb{N} \setminus {}^{st}\mathbb{N})$ and suppose that $n \approx \infty$. Then $n-1$ is also unlimited, therefore, $n-1 \in \mathbb{N} \setminus {}^{st}\mathbb{N}$, contrary to the definition of n. But the inequality $n \ll \infty$ is also impossible because $n \ll \infty \implies st(n)$, contrary to $n \in \mathbb{N} \setminus {}^{st}\mathbb{N}$. ►

Exercise 3
Prove that ${}^{st}\mathbb{N}$ is strictly external. Hint: ${}^{st}\mathbb{N}$ is bounded by each $\omega \approx \infty$, but $\max({}^{st}\mathbb{N})$ does not exist.

Remark 5
With the help of the word "standard," we have constructed not only a new kind of numbers but also a new kind of sets, namely strictly external ones. This once more presents the power of words which are happily selected.

Warning 1
Nelson's IST knows only internal sets: in IST "a set" means "an internal set." Thus, the Internal Set Theory does not know what an external set is. For instance, in IST there is no ${}^{st}\mathbb{N}$, ${}^{st}\mathbb{Q}$, ${}^{st}\mathbb{R}$, and so on. What is the ground for manipulation with such (and other) external sets? The matter is that IST knows external formulas. In IST, we don't understand what ${}^{st}\mathbb{N}$ is, but we understand what $x \in {}^{st}\mathbb{N}$ means. $x \in {}^{st}\mathbb{N}$ is just an abbreviation for $x \in \mathbb{N} \wedge st(x)$. Another example: let A, B be standard sets, and $p(x)$, $q(x)$ be external formulae. External sets $A_p := \{x \in A : \ p(x)\}$ and $B_q := \{x \in B : \ q(x)\}$ are outside of IST. Therefore, $A_p \cup B_q$ cannot be formed in IST. But the formula $x \in A_p \cup B_q$ makes sense. It is an abbreviation of $(x \in A \wedge p(x)) \vee (x \in B \wedge q(x))$.

Exercise 4
What about $A_p \cap B_q$, $A_p \setminus B_q$, $A_p \times B_q$, $B_q^{A_p}$? Let A be external. What is $\{A\}$?

Exercise 5
For any $n \in \mathbb{N}$, denote by $f(n)$ the first $k \in \mathbb{N}$, such that the interval $]\frac{k-1}{n}, \frac{k}{n}]$ contains a standard real number. Why is this definition of the function f not correct in the framework of IST?

7. Formulae

Mathematical formulae express mathematical statements. For instance, formulae "$5 + 3 = 8$", "$2 < 1$" are sentences (of Arithmetics), the former is *true*, the latter is *false*. The formulae "$x + 3 = 8$", "$x < 1$" are not sentences, for they are not false or true. But they become sentences if we replace x by a fixed real number. Formulae such as "$x + 3 = 8$", "$x < 1$" are *predicates*, they express some *property* (for example "$x < 1$" is the property of real numbers "to be less than 1"). The variable x, which occurs in a formula $p(x)$ in which after it is replaced by an (admissible) constant, it results a sentence (true or false), is said to be *free*. Thus, x is a free variable in "$x+3 = 5$". If a formula is denoted by $p(x, y)$, it means that there are two free variables. To get a sentence, we ought to replace both x and y

by (admissible) constants. Evidently, the quantity of free variables can be larger. For instance, in the equation $3x - 2y + 5z = 1$ (of a plane in the space $\mathbb{R}^3$) we have three free variables.

From the formula $p(x)$, one can obtain a sentence (true or false) not only by replacing x by a constant, but also by *bounding* it by the quantifier $\forall\ \equiv$ "for all", or $\exists\ \equiv$ "for some". Thus, $(\forall x \in \mathbb{R})(x+3=8)$ and $(\exists x \in \mathbb{R})(x+3=8)$ contain x as a *bounded* (apparent) variable. They are sentences: the first is false, the second is true.

In NSA, we deal with *internal* and *external* formulae. The basic external formula is $st(x)$ (read "x is standard" for $st(x)$). Each formula, in which "$st(x)$" occurs explicitly or implicitly, is also said to be *external*. *Internal* formulae are those in which "$st(x)$" does not occur in any way (explicitly or implicitly). Obviously, we only mean formulae which are well–constructed by the laws of Mathematics and Logic.

Let us consider some examples. The formulae $x \approx 0$, $|x| \ll \infty$, $x \gg 0$ are all external. For instance, $|x| \ll \infty$ is an abbreviation for $(x \in \mathbb{R}) \wedge (\exists n \in \mathbb{N}) st(n) \wedge (|x| < n)$ which contains "st(x)." At the same time, the formulae $x = 0$, $x > 0$, $|x| < 1$ are internal. Let a, b be straight lines. Then, $a \parallel b$, $a \perp b$ are internal sentences. If values of variables x, y are straight lines, then $x \perp a$, $x \parallel a$, $x \perp y$, $x \parallel y$ are internal formulae.

A formula can contain variables and constants. For instance, the formula $a < x < y < b$ contains two variables x, y, and two constants a, b; traditionally, variables are denoted by $x, y, z, \ldots, u, v, w, \ldots$ and constants by $a, b, c, \ldots, k, l, \ldots$; $m, n, p, \ldots$ are reserved for variables which range over the set $\mathbb{N}$ of natural numbers.

We accept the following **definition**: An internal formula containing only standard constants is said to be a *standard formula*. For instance, the formula $0 < x < \varepsilon$ is internal. It is standard if ε denotes a standard number, but if ε denotes e.g. an infinitesimal number, it is non standard.

8. The transfer principle

According to Nelson's approach to NSA (the main work of Nelson on this subject is IST = Internal Set Theory = Idealization + Standardization + Transfer; (see Lyantse, Kudryk, 1997) we need to add only three new axioms to OM (based on ZFC). One of them is *the transfer principle* (T). We express it by using the following statement:

$$(\exists\, x)\, p(x) \implies \exists^{st} x\ p(x); \tag{T}$$

here $p(x)$ denotes an arbitrary *standard* formula. Therefore, $p(x)$ in (T) does not involve $st(x)$ as a subformula, and all constants in $p(x)$ are standard. The domain of action of the quantifier $\exists$ in (T) is the universe $\mathbf{I}$ of internal sets. Thus, the principle (T) can be rewritten in the following form:

$$(\exists\, x \in \mathbf{I}) p(x) \implies (\exists x \in \mathbf{S}) p(x).$$

Since $\mathbf{S} \subset \mathbf{I}$, we can replace " $\implies$ " with " $\iff$ ". Therefore, (T) is, in essence, the statement

$$(\exists x \in \mathbf{I})\ p(x) \iff (\exists x \in \mathbf{S})\ p(x). \tag{T}$$

Let us explain the content of (T) in other words. Let $p(x)$ be a well–defined standard formula with a free variable x which ranges over $\mathbf{I}$. Let $dom_t p$ be a part of $\mathbf{I}$, such that $p(x)$ is true for $x \in dom_t p$. (T) says that *if $dom_t p$ is not empty then $dom_t p \cap \mathbf{S}$ is not empty* as well.

8.1. The characterization of standardness

Now we can repeat in a more rigorous way what we have said in Section 2. Remember that one reads "$\exists! x$" as "there is a unique x." "$\exists!^{st} x$" means "there is a unique standard x." Let us formulate the principle (Ch) (characterization of standardness):

$$(\exists! x \in \mathbf{I})\ p(x)\ \wedge\ p(x_0) \implies st(x_0). \tag{Ch}$$

Indeed, *if there exists a unique (internal) x for which the standard formula $p(x)$ is true,* then, by (T), *it must be standard.*

Let us consider some examples.

Example 1
The empty set $\emptyset$ is standard.

The formula $p(x) \equiv (\forall y)(y \notin x)$ is standard (it contains no constant). The statement $p(\emptyset)$ is true: $(\forall y)(y \notin \emptyset)$. This defines $\emptyset$ uniquely. To prove it, let us recall the extensionality principle of OM:

$$(A = B) \iff (\forall x)(x \in A \iff x \in B).$$

Now, if both $\emptyset_1$ and $\emptyset_2$ are empty, then for any x both $x \in \emptyset_1$ and $x \in \emptyset_2$ are false. Therefore, the formula $x \in \emptyset_1 \iff x \in \emptyset_2$ is true.

Example 2
If A is standard, then so is $B := \{A\}$.

Really, consider the formula $p(x) = (\forall y)(y \in x \Leftrightarrow y = A)$. This formula is standard, because A is standard, and B is a unique set, for which $p(x)$ is true.

In the same way we can show that

Example 3
If $A, B, C, \ldots$ are standard, then so are $\{A\}$, $\{A, B\}$, $\{A, B, C\} \ldots$.

Example 4
If A, B are standard, then the ordered pair (A, B) is standard.

Indeed, according to Kuratowski, $(A, B) = \{\{A\}, \{A, B\}\}$.

Example 5
If A, B are standard, then $A \cup B$, $A \cap B$, $A \setminus B$, $A \times B$ are standard.

For instance, $A \times B$ is a *unique* set, for which $p(A \times B)$ is true, where $p(x)$ is the following *standard* formula, $(\forall y)(y \in x) \iff (\exists a \in A)(\exists b \in B)(y = (a, b))$.

EXAMPLE 6
The numbers $0, 1, 2, 3, \ldots$ are standard.

This follows from Example 1 to Example 4, because $0 := \emptyset$, $1 := \{0\}$, $2 := \{0, 1\}$, $3 := \{0, 1, 2\} \ldots$

EXAMPLE 7
The set $\mathbb{N}$ of natural numbers is standard.

Indeed, $\mathbb{N}$ is a unique x which satisfies the standard formula $p(x) \wedge q(x) \wedge r(x)$, where $p(x) \equiv (1 \in x)$, $q(x) \equiv (\forall y)(y \in x \implies y \cup \{y\} \in x)$, $r(x) \equiv (\forall z)(p(z) \wedge q(z) \implies x \subseteq z)$.

REMARK 6
Instead of $y \cup \{y\}$, one usually writes $y + 1$; $r(x)$ means that x is a minimal set which satisfies $p(x)$ and $q(x)$.

EXERCISE 6
Prove that the set $\mathbb{Z}$ of all whole numbers is standard.

EXAMPLE 8
Let A be a standard set, and $p(x)$ a standard formula. Then $B := \{x \in A : \ p(x)\}$ is standard.

B is uniquely determined by the standard formula $p(x) \equiv (\forall y) \quad (y \in x \iff y \in A \wedge p(y))$.

EXAMPLE 9
The set $\mathbb{Q}$ of all rational numbers is standard.

Really, any rational number can be *uniquely* represented as y/x, where $x \in \mathbb{N}$, $y \in \mathbb{Z}$, and x, y have no common divisors. Thus, $\mathbb{Q}$ is a part of $\mathbb{N} \times \mathbb{Z}$ defined by a standard condition.

EXAMPLE 10
If A, B are standard, then the set B^A of all functions f such that $\operatorname{dom} f = A$, $\operatorname{im} f \subseteq B$, is standard.

Indeed, B^A is a part of the standard set $A \times B$ which consists of all f satisfying the standard formula $f \subseteq A \times B \wedge \forall x \in A \ \exists! \, y \in B \ (x, y) \in f$.

EXERCISE 7
1° Prove that the set $\mathbb{R}$ of all real numbers is standard. *Hint*: represent $\mathbb{R}$ as $\{r \in \mathbb{Q}^{\mathbb{N}} : \ p(r)\}$, where $p(r)$ is a suitable standard formula.

2° Let A be a standard bounded set of real numbers. Prove that $\inf A$ and $\sup A$ are standard.

3° Let $f \in \mathbb{R}^{\mathbb{R}}$ be a standard function. Suppose that the equation $f(x) = 0$ has only a finite number of roots. Prove that each such root is standard. Hint: apply (I_1) to $\{x \in \mathbb{R} : \ f(x) = 0\}$. Does the equation $\sin x = 0$ have non-standard roots?

4° If E is a standard finite set, then $\operatorname{card} E$ is a standard natural number. Why?

5° Let A, B be standard sets, and $A \cap B \neq \emptyset$. Prove that A and B have a common standard element. Prove that the common point of two standard straight lines is standard.

6° Consider the Cauchy problem $y' = f(x, y)$, $y_{|x=x_0} = y_0$. Provide a condition for its solution to be standard.

Proposition 4

1° *A standard function takes a standard value at any standard point.*

2° *The inverse image of a standard set under a standard transformation is standard.*

Proof is left as an exercise for the reader.

9. The second form of the transfer principle

Apply to the principle (T) the following law of logic $(a \Leftrightarrow b) \iff (\neg a \Leftrightarrow \neg b)$. Note that if the formula $p(x)$ is standard, then so is $\neg p(x)$. This way, we get

$$\forall^{st} x \ p(x) \iff \forall x \, p(x), \tag{T'}$$

where $p(x)$ is an *arbitrary standard* formula. This (T') is the second form of the transfer principle. More strictly, (T') is as follows:

$$(\forall x \in \mathbf{S}) \ p(x) \iff (\forall x \in \mathbf{I}) \ p(x). \tag{T'}$$

Thus, *if each standard x has a standard property p, then each internal x has this property.* In other words, as long as we consider standard properties only, we cannot differentiate standard objects from non-standard (internal) ones. Standard properties are their common properties.

Now let us consider some applications.

9.1. Non-standard extensionality principle

It states that *a standard set A is uniquely determined by the totality ${}^{st}A$ of its standard elements.* Namely,

$$(\forall A, B \in \mathbf{S}) \ (A = B \iff {}^{st}A = {}^{st}B).$$

(We recall that ${}^{st}A$ is the external set $\{x \in A : \ st(x)\}$).

Proof. By the extensionality principle of OM, $A = B$ if (and only if) $\forall x \ p(x)$, where $p(x) \equiv (x \in A) \iff (x \in B)$. If A and B are standard, this $p(x)$ is standard. (T') tells us that $\forall^{st} x \ p(x)$ is sufficient for $A = B$, i.e. ${}^{st}A = {}^{st}B$ is sufficient for $A = B$. ►

Exercise 8

1° Prove that if A, B are standard, then $A \subset B$, whenever ${}^{st}A \subset {}^{st}B$.

2° Give a counterexample for $({}^{st}A = {}^{st}B) \implies (A = B)$ with non-standard A, B.

9.2. The uniqueness of standard functions

Let X, Y be standard sets and $f \in Y^X$ be a standard function. Such f *is uniquely determined by its values at standard points.* To be more exact, if $f_1,\ f_2 \in {}^{st}(Y^X)$, then

$$(\forall^{st} x \in X)\ (f_1(x) = f_2(x)) \implies (\forall x \in X)\ (f_1(x) = f_2(x)). \tag{1}$$

Proof. If f_1, f_2 are standard, then $f_1(x) = f_2(x)$ is a standard formula, and, by (T'), we have (1). ►

Corollary 2

A standard sequence $u \in \mathbb{R}^{\mathbb{N}}$ can be defined in no way as follows

$$u_n = \begin{cases} f(n) & for\ n \ll \infty, \\ g(n) & for\ n \approx \infty, \end{cases}$$

where f, g are different standard functions.

Remark 7

The transfer principles may be extended to formulae with an arbitrary quantity of free variables. To indicate explicitly the standardness of these formulae, we write

$$\begin{gathered} \forall^{st} t_1 \cdots \forall^{st} t_r \exists x_1 \cdots \exists x_n\ p(x_1, \ldots, x_n, t_1, \ldots, t_r) \implies \\ \implies \forall^{st} t_1 \cdots \forall^{st} t_r \exists^{st} x_1 \cdots \exists^{st} x_n\ p(x_1, \ldots, x_n, t_1, \ldots, t_r), \end{gathered} \tag{T}$$

$$\begin{gathered} \forall^{st} t_1 \cdots \forall^{st} t_r \forall^{st} x_1 \cdots \forall^{st} x_n\ p(x_1, \ldots, x_n, t_1, \ldots, t_r) \implies \\ \implies \forall^{st} t_1 \cdots \forall^{st} t_r \forall x_1 \cdots \forall x_n\ p(x_1, \ldots, x_n, t_1, \ldots, t_r), \end{gathered} \tag{T'}$$

where $p(x_1, \ldots, x_n, t_1, \ldots, t_r)$ is an arbitrary *internal* formula which contains no constants, with free variables $x_1, \ldots, x_n, t_1, \ldots t_r$.

9.3. The uniqueness of a standard relation

Let X, Y be standard sets, and $\mathcal{R}_1,\ \mathcal{R}_2$ be standard relations between elements of X and Y. (This means that $\mathcal{R}_i \subseteq X \times Y$, $i = 1, 2$; as usually we write $x\mathcal{R}_i y$ for $(x, y) \in \mathcal{R}_i$).) Then

$$\begin{aligned} &(\forall^{st}(x, y) \in X \times Y)(x\mathcal{R}_1 y \iff x\mathcal{R}_2 y) \\ &\qquad\qquad \iff (\forall (x, y) \in X \times Y)(x\mathcal{R}_1 y \iff x\mathcal{R}_2 y). \end{aligned}$$

Proof is left as an exercise for the reader.

Example 11
Define $\mathcal{R} \subset \mathbb{R}^2$ by $x\mathcal{R}y \equiv x = y$ for standard x, y and $x\mathcal{R}y \equiv x \leq y$ for non-standard x, y. Then, $\mathcal{R}$ is not standard.

Example 12
We have $e^x > 0$, $\cos(x+y) = \cos x \cos y - \sin x \sin y$ for all $x, y \in \mathbb{R}$, because these formulae are standard and true for $x, y \in {}^{st}\mathbb{R}$.

10. The special standardization principle (S_0)

The principle (I_0) informs us that we cannot create any standard infinite set without non-standard elements. The standardization principle tells us that sometimes it is possible to obtain something standard from something nonstandard. It is the third axiom of Nelson's IST. Now we formulate only some of its corollaries, namely the special standardization principle (S_0). It is as follows:

Let x be a limited real number. There exists *a unique standard* real number y such that $x \approx y$. Thus,

$$(\forall x \in \mathbb{R})(\|x\| \ll \infty \implies (\exists!^{st} y \in \mathbb{R})(x \approx y)). \qquad (S_0)$$

The number y in (S_0) is denoted by $^\circ x$ and said to be *the shadow* (or *the standard part*) of x. So the shadow of $x \in \mathbb{R}$, $|x| \ll \infty$, is defined *uniquely* by

$$^\circ x \in {}^{st}\mathbb{R} \;\; and \;\; {}^\circ x \approx x.$$

The uniqueness of the shadow immediately implies the following proposition.

Proposition 5
1° *If $x \in \mathbb{R}$ is standard, then $^\circ x = x$.*

2° *The unique standard infinitesimal is zero:*

$$x \in {}^{st}\mathbb{R} \implies {}^\circ x = x; \quad x \in {}^{st}\mathbb{R} \wedge x \approx 0 \implies x = 0.$$

3° *If $x, y \in {}^{st}\mathbb{R}$, and $x \approx y$, then $x = y$.*

Let us define

$$\mathbb{F} := \{x \in \mathbb{R} : \; |x| \ll \infty\}, \quad \mathbb{I} := \{x \in \mathbb{R} : \; x \approx 0\}.$$

Note that the set $\mathbb{F}$ of all limited real numbers and the set $\mathbb{I}$ of all real infinitesimals are *strictly external.* Indeed, $\mathbb{F}$ and $\mathbb{I}$ are non empty, $\mathbb{F}$ is bounded by every positive unlimited $\omega \in \mathbb{R}$, $\mathbb{I}$ is bounded, e.g., by 10^{-10}, but $\sup \mathbb{F}$, $\sup \mathbb{I}$, $\inf \mathbb{F}$, $\inf \mathbb{I}$ do not exist.

The immediate corollary from (S_0) is as follows:

$$\forall x \in \mathbb{F} \quad x = {}^\circ x + {}^i x, \quad {}^\circ x \in {}^{st}\mathbb{R}, \quad {}^i x \in \mathbb{I}. \qquad (2)$$

The uniqueness of the shadow implies that this decomposition is unique. In other words, $\mathbb{F}$ is a *direct sum*:

$$\mathbb{F} = {}^{st}\mathbb{R} \dot{+} \mathbb{I}.$$

11. $\mathbb{R}$, ${}^{st}\mathbb{R}$, $\mathbb{F}$, and $\mathbb{I}$.

In OM, the set $\mathbb{R}$ of all real numbers together with the arithmetic operations $+$ and $\cdot$ and with the order relation $\leq$ is defined as a complete linearly ordered field. It is archimedean, i.e.

$$(\forall x \in \mathbb{R})(x \neq 0 \implies (\exists n \in \mathbb{N})(n|x| > 1)). \tag{3}$$

If x in (3) is infinitesimal, then the corresponding n must be infinite. But from (I_0) we can conclude that

$$(\forall^{st} x \in \mathbb{R})(x \neq 0 \implies (\exists^{st} n \in \mathbb{N})(n|x| > 1)).$$

An analogous *remark* also concerns other statements about $\mathbb{R}$. For instance, consider the Borel–Lebesgue lemma. Let $\mathcal{U} = \{\mathcal{U}_\alpha\}$ be an open cover of some bounded closed set $E \subset \mathbb{R}$. The lemma states that there exists a finite subcover $\mathcal{U}' = \{\mathcal{U}_{\alpha_1}, \ldots, \mathcal{U}_{\alpha_n}\}$. This is true even if the exact bounds of E are infinite, and the length of a $\mathcal{U}_\alpha$ is ≈ 0. Obviously, in this case we have $n \approx \infty$. But if E and $\mathcal{U}$ are standard, then (I_0) implies that $\mathcal{U}'$ can be chosen as standard.

The set ${}^{st}\mathbb{R} := \{x \in \mathbb{R} :\ st(x)\}$ is a subfield of the field $\mathbb{R}$. For instance, we have $x \in {}^{st}\mathbb{R} \wedge x \neq 0 \implies x^{-1} \in {}^{st}\mathbb{R}$, as a standard function (here $x \mapsto x^{-1}$) takes standard values at standard points. ${}^{st}\mathbb{R}$ is strictly external for the same reasons as $\mathbb{F}$.

Theorem 1

1° *The (strictly external) set $\mathbb{F}$ is a linearly ordered ring, a subring of $\mathbb{R}$. This means that*

$$\mathbb{F} \subset \mathbb{R}, \quad (\forall x, y \in \mathbb{F})\ (x + y \in \mathbb{F} \wedge xy \in \mathbb{F})$$
$$(\forall x \in \mathbb{R})\ (\forall y \in \mathbb{F})\ (|x| \leq y \implies x \in \mathbb{F}).$$

2° *The (strictly external) set $\mathbb{I}$ is an ideal of the linearly ordered ring $\mathbb{F}$. This means that*

$$\mathbb{I} \subset \mathbb{F},\ (\forall x, y \in \mathbb{I})\ (x + y \in \mathbb{I}),\ (\forall x \in \mathbb{I})\ (\forall y \in \mathbb{F})\ (xy \in \mathbb{I}),$$
$$(\forall x \in \mathbb{F})(\forall y \in \mathbb{I})(|x| \leq y \implies x \in \mathbb{I}).$$

3° *The map $x \mapsto^\circ x$ is a (strictly external) homomorphism of the ring $\mathbb{F}$ onto the field ${}^{st}\mathbb{R}$. This means that*

$$(\forall x, y \in \mathbb{F})({}^\circ(x + y) = {}^\circ x + {}^\circ y, \quad {}^\circ(xy) = {}^\circ x \cdot {}^\circ y),$$
$$(x \leq y \implies {}^\circ x \leq {}^\circ y), \quad (\forall x \in \mathbb{F})(\exists! \xi \in {}^{st}\mathbb{R})(\xi =^\circ x).$$

Proof is left as an exercise for the reader. For instance, let $x \in \mathbb{F}$, $y \in \mathbb{I}$. Then $|xy| < mn^{-1}$ for some $m \in {}^{st}\mathbb{N}$ and all $n \in {}^{st}\mathbb{N}$. If we take $n = km$, then we obtain $|xy| < k^{-1}$ with an arbitrary $k \in {}^{st}\mathbb{N}$, i.e. $xy \in \mathbb{I}$. Next, let $x, y \in \mathbb{F}$. According to (2), we have $xy = {}^\circ x {}^\circ y + \alpha$, where $\alpha := {}^i x {}^\circ y + {}^\circ x {}^i y + {}^i x {}^i y \approx 0$, hence ${}^\circ(xy) = {}^\circ x {}^\circ y$. Suppose that $x, y \in \mathbb{F}$, $x < y$, but ${}^\circ x > {}^\circ y$. Since ${}^\circ x + {}^i x < {}^\circ y + {}^i y$, we have ${}^\circ x - {}^\circ y < {}^i y - {}^i x$. This is a contradiction, because ${}^\circ x - {}^\circ y$ is a positive standard, and ${}^i y - {}^i x \approx 0$. And so on.

Remark 8
The map $x \mapsto {}^\circ x$ is *idempotent*, i.e. $\forall x \in \mathbb{F}\ {}^\circ({}^\circ x) = {}^\circ x$. Its *kernel* $\{x \in \mathbb{F} :\ {}^\circ x = 0\}$ is the ideal $\mathbb{I}$ of infinitesimal numbers. We can say that $x \mapsto {}^\circ x$ is *a projection* of $\mathbb{F}$ onto ${}^{st}\mathbb{R}$ parallelly to $\mathbb{I}$. Clearly, $x \mapsto {}^i x$ is *a projection* of $\mathbb{F}$ onto $\mathbb{I}$, parallelly to (its kernel) ${}^{st}\mathbb{R} = \{x \in \mathbb{F} :\ {}^i x = 0\}$.

At the very end, some "exotic" examples. A finite set E is said to be *hyperfinite* if $\operatorname{card} E \approx \infty$.

Example 13
An external "finite" set which contains the "infinite" set ${}^{st}\mathbb{R}$.

Take an $\omega \in \mathbb{N} \setminus {}^{st}\mathbb{N}$ and form a sequence of segments $\Delta_k := [k/\omega, (k+1)/\omega[$, where $k = -\omega^2$, $-\omega^2 + 1$, …, $\omega^2 - 2$, $\omega^2 - 1$. Note that

$$\bigcup_{k=-\omega^2}^{\omega^2-1} \Delta_k = [-\omega, \omega[\supset {}^{st}\mathbb{R}.$$

None of Δ_k contains more than one standard point. Indeed, the length of Δ_k is $1/\omega \approx 0$, therefore if $x_1, x_2 \in \Delta_k$, then $x_1 \approx x_2$ and $x_1 = x_2$ whenever both x_1 and x_2 are standard. It is evident that only some Δ_k contain a standard point and never more than one. Denote by E' the set of centers of such Δ_k which do not contain a standard point, and put $E = E' \cup {}^{st}\mathbb{R}$. We see that E is a totality of $2\omega^2 \in \mathbb{N}$ points. It contains ${}^{st}\mathbb{R}$, but, unfortunately, it is defined *externally*.

The existence of an *internal* E such that ${}^{st}\mathbb{R} \subset E$ and $\operatorname{card} E \in \mathbb{N}$ can be shown. Obviously, $\operatorname{card} E \approx \infty$ for such E.

12. On decimal fractions

By the transfer principle (T'), each $x \in \mathbb{R}$ (standard or not) can be represented as

$$x = x_0, x_1 x_2 x_3 \ldots, \tag{4}$$

where $x_0 \in \mathbb{Z}$, $\forall n \in \mathbb{N}\ \ x_n \in \{0, 1, \ldots, 9\}$. Recall that (4) means that

$$\forall n \in \mathbb{N} \quad x_0 + \frac{x_1}{10} + \cdots + \frac{x_n}{10^n} \le x \le x_0 + \frac{x_1}{10} + \cdots + \frac{x_n}{10^n} + \frac{1}{10^n}. \tag{5}$$

By Cantor's principle of *nested segments*, (5) determines x uniquely. For this, it is necessary for the sequence $(x_n)_{n\in\mathbb{N}}$ in (4) to be internal: if $(x_n)_{n\in\mathbb{N}}$ is strictly external, then formula (4) represents no number. For instance, the fraction $0,000\ldots\ \ldots 999\ldots$ (0s are at finite positions, 9s are at infinite ones) is not a real number. Obviously, *x in (4) is standard if the sequence $(x_n)_{n\in\mathbb{N}}$ is standard.*

Proposition 6
The number $x \in \mathbb{R}$ is a positive infinitesimal if and only if its decimal expansion has the form

$$x = 0, 0 \ldots 0 x_{\omega+1} x_{\omega+2} \ldots, \tag{6}$$

where $x_{\omega+1} \ne 0$ and $\omega \approx \infty$.

Proof. From (5) and (6) we conclude that

$$\frac{x_{\omega+1}}{10^{\omega+1}} \le x \le \frac{x_{\omega+1}+1}{10^{\omega+1}},$$

where $0 < x_{\omega+1} \le 9$. Therefore, if $\omega \ll \infty$, then $x > 10^{-\omega-1} \gg 0$, and if $\omega \approx \infty$, then $x < 10^{-\omega-1} \approx 0$. ▶

Corollary 3
Let $x \in \mathbb{R}$ be positive. According to (2), we have $x = y + z$, where $y = {}^\circ x$ is the standard part (shadow) of x, and $z = {}^i x$ is its infinitesimal part. Let $y_0, y_1 y_2 \dots$ be the decimal expansion of y. Then, for an infinite natural ω we have

$$x_0 = y_0, \quad x_1 = y_1, \dots, x_\omega = y_\omega. \tag{7}$$

Indeed, since $z \approx 0$, we have $z = 0,0\dots z_{\omega+1}z_{\omega+2}\dots$ for some $\omega \approx +\infty$. But $y = x - z$ and the substraction here can only change one decimal in x before $x_{\omega+1}$. ▶

Remark 9
Let $x = x_0, x_1 x_2 \dots$ In order to find the shadow ${}^\circ x$, we only have to know x_n for $n \in {}^{st}\mathbb{N}$. Indeed, from (7) it follows that

$$\forall n \in {}^{st}\mathbb{N} \quad ({}^\circ x)_n = {}^\circ x_n.$$

On the other hand, *a standard* sequence $(a_n)_{n\in\mathbb{N}}$ is uniquely determined by a_n, $n \in {}^{st}\mathbb{N}$ (see Subsection 9.2, Corollary 2).

13. Some permanence principles. Robinson's lemma

An (internal) set $\tilde{N} \subseteq \mathbb{N}$ is said to be *modest* if

$$\exists\, m \ll \infty \quad \tilde{N} \subset \{1, 2, \dots, m\},$$

and it is said to be *greedy* if

$$\exists\, \omega \approx \infty \quad \{1, 2, \dots, \omega\} \subset \tilde{N}.$$

The simplest permanence principles are the following statements.

Proposition 7
Let $\tilde{N} \in 2^{\mathbb{N}}$, then

$$\tilde{N} \subset {}^{st}\mathbb{N} \implies \tilde{N} \text{ is } \text{ modest}; \tag{8}$$

$${}^{st}\mathbb{N} \subset \tilde{N} \implies \tilde{N} \text{ is } \text{ greedy}. \tag{9}$$

Proof. Let $\tilde{N} \subset {}^{st}\mathbb{N}$, denote $N' := \{m \in \mathbb{N} : \ \tilde{N} \subset \{1, \dots, m\}\}$. We have $\mathbb{N} \setminus {}^{st}\mathbb{N} \subseteq N'$. But N' is internal, and $\mathbb{N} \setminus {}^{st}\mathbb{N}$ is strictly external. Therefore, $N' \ne \mathbb{N} \setminus {}^{st}\mathbb{N}$, i.e., $\exists\, m \ll \infty \ \ m \in N'$, and (13.1) holds.

Now let ${}^{st}\mathbb{N} \subset \tilde{N}$. Put $N' = \{\omega \in \mathbb{N} : \ \{1, \dots, \omega\} \subset \tilde{N}\}$. By the above arguments, N' is internal. ${}^{st}\mathbb{N} \subseteq N'$, ${}^{st}\mathbb{N}$ is strictly external, therefore, N' contains some $\omega \approx \infty$. For this ω we have $\{1, \dots, \omega\} \subseteq \tilde{N}$. ▶

Corollary 4
Let $\tilde{N} \in 2^{\mathbb{N}}$, *then*

$$\tilde{N} \subset \mathbb{N} \setminus {}^{st}\mathbb{N} \implies \exists\, \omega \approx \infty \;\; \tilde{N} \subset \{\omega+1, \omega+2, \ldots\}, \tag{10}$$

$$\mathbb{N} \setminus {}^{st}\mathbb{N} \subset \tilde{N} \implies \exists\, m \ll \infty \;\; \{m+1, m+2, \ldots\} \subset \tilde{N}. \tag{11}$$

Proof. If $\tilde{N} \subset \mathbb{N} \setminus {}^{st}\mathbb{N}$, then ${}^{st}\mathbb{N} \subset \mathbb{N} \setminus \tilde{N}$, therefore, $\mathbb{N} \setminus \tilde{N}$ is greedy: $\exists\, \omega \approx \infty$ $\{1, \ldots, \omega\} \subset \mathbb{N} \setminus \tilde{N}$, hence (10) holds. (11) can be proven in an analogous way. ▶

A remarkable permanence principle is the following famous

Lemma 1 (Robinson's lemma)
Let $x \in \mathbb{R}^{\mathbb{N}}$ *and* $\forall\, n \in {}^{st}\mathbb{N} \;\; x_n \approx 0$. *Then* $\exists\, \omega \approx \infty \;\; \forall\, n \le \omega \;\; x_n \approx 0$.

Proof. Let $\tilde{N} := \{n \in \mathbb{N} : \; n|x_n| < 1\}$. Since ${}^{st}\mathbb{N} \subset \tilde{N}$, $\tilde{N}$ is greedy. Let $\omega \approx \infty$ be such that $\{1, \ldots, \omega\} \subset \tilde{N}$. Then $\forall\, n \le \omega \;\; |x_n| < n^{-1}$, therefore, $\forall\, n \le \omega \;\; x_n \approx 0$. ▶

The other permanence principles are as follows.

Proposition 8
Let $E \in 2^{\mathbb{R}}$, *then*

$$\begin{aligned}
E \subset \mathbb{I} &\implies \exists\, \varepsilon \approx 0 \;\; E \subset [-\varepsilon, \varepsilon] \quad \text{(modesty)}\,,\\
\mathbb{I} \subset E &\implies \exists\, \varepsilon \gg 0 \;\; [-\varepsilon, \varepsilon] \subset E \quad \text{(greediness)}\,,\\
E \subset \mathbb{F} &\implies \exists\, a \ll \infty \;\; E \subset [-a, a] \quad \text{(modesty)}\,,\\
\mathbb{F} \subset E &\implies \exists\, a \approx \infty \;\; [-a, a] \subset E \quad \text{(greediness)}\,,
\end{aligned}$$

where, as before, $\mathbb{I} := \{x \in \mathbb{R} : \; x \approx 0\}$, $\mathbb{F} := \{x \in \mathbb{R} : \; |x| \ll \infty\}$.

Proof. Let $\mathbb{I} \subset E$. Set $\tilde{N} := \{n \in \mathbb{N} : \; \forall\, k \le n \;\; |x| < k^{-1} \implies x \in E\}$. We have $\mathbb{N} \setminus {}^{st}\mathbb{N} \subset \tilde{N}$. By (11), $\tilde{N}$ contains some $m \ll \infty$. Therefore, $|x| \le m^{-1} \implies x \in E$. We left the rest of the proof as an exercise for the reader. ▶

14. Applications to sequences

14.1. The limit of a sequence

Let $x \in \mathbb{R}^{\mathbb{N}}$ (read "x is a sequence of real numbers"; usually we write x_n instead of $x(n)$). We assume that the reader is familiar with the ordinary definition of convergence and the limit of x:

$$\begin{aligned}
&x \text{ is convergent (or Cauchy)} \equiv \; (\forall \varepsilon > 0)\, (\exists N \in \mathbb{N})\, (p, q > N \implies |x_p - x_q| < \varepsilon),\\
&\lim_{n\to\infty} x_n = \ell \equiv (\forall \varepsilon > 0)(\exists N \in \mathbb{N})(\forall n > N)\; |x_n - \ell| < \varepsilon.
\end{aligned}$$

It turns out that, for standard sequences, this definition can be simplified, according to the aspirations of naturalists.

Theorem 2
Let $x \in {}^{st}(\mathbb{R}^{\mathbb{N}})$, *then* x *is convergent if and only if*

$$\forall\; p, q \approx \infty \quad x_p \approx x_q. \tag{12}$$

Let $x \in {}^{st}(\mathbb{R}^{\mathbb{N}})$ be convergent. Then, for $n \approx \infty$ we have $|x| \ll \infty$, and therefore x_n has the shadow $\ell = {}^{\circ}x_n$, which is the same for all $n \approx \infty$. It is the limit of x:

$$\lim_{n\to\infty} x_n = {}^{\circ}(x_\omega) \quad for\ \omega \approx \infty.$$

Proof. Let x be a standard convergent sequence of reals x_n and $\lim\limits_{n\to\infty} x_n = \ell$. This ℓ is standard by the transfer principle (T). Take an arbitrary $\varepsilon > 0$ and find $N \in \mathbb{N}$ such that $|x_n - \ell| < \varepsilon$ for $n > N$. Once again, by (T), this N can be chosen as standard. Let $n \in \mathbb{N} \setminus {}^{st}\mathbb{N}$. We now have $n > N$. Therefore, $|x_n - \ell| < \varepsilon$. As ε is arbitrary, we have $|x_n - \ell| \approx 0$. We see that $\forall\, n \approx \infty \;\; {}^{\circ}(x_n) = \ell$. Also we know that $\forall\, p, q \approx \infty \;\; |x_p - x_q| \le |x_p - \ell| + |\ell - x_q| \approx 0$, that is $x_p \approx x_q$.

Conversely, assume that (12) is satisfied. Consider the set $\tilde{N} := \{n \in \mathbb{N} : |x_n - x_\omega| < 1\}$, where ω is a fixed unlimited natural number. Since the formula $p(n) \equiv |x_n - x_\omega| < 1$ is internal, the set $\tilde{N}$ is internal, too. By (12), $\mathbb{N} \setminus {}^{st}\mathbb{N} \subseteq \tilde{N}$. By (11), $\tilde{N}$ contains some *standard* n_ω. Hence (because x_{n_ω} is standard) $|x_\omega| \le |x_{n_\omega}| + 1 \ll \infty$, so x_ω has the shadow $\ell :=^{\circ} (x_\omega)$. By (12), $\forall\, n \approx \infty \;\; x_n \approx \ell$. Now take an arbitrary standard positive ε. Note that the sentence $(\exists\, N \in \mathbb{N})\ (\forall\, n > N)\ \ (|x_n - \ell| < \varepsilon)$ is true. Indeed, it is sufficient to choose $N \approx \infty$, and then $n > N \implies |x_n - \ell| < \varepsilon$. Now applying the transfer principle (T'), we get: $(\forall\, \varepsilon > 0)\ (\exists\, N \in \mathbb{N})\ (\forall\, n > N)\ (|x_n - \ell| < \varepsilon)$. ▶

Example 14

1° The sequence $(1/n)_{n\in\mathbb{N}}$ and the number 0 are standard. For $n \approx \infty$, we have $1/n \approx 0$. Therefore, $\lim\limits_{n\to\infty} 1/n = 0$.

2° Let $\varepsilon \neq 0$ be infinitesimal. For $n \approx \infty$, we have $1/n \approx \varepsilon$. But $\lim\limits_{n\to\infty} 1/n \neq \varepsilon$. This fact does not contradict theorem 2, because ε is not standard.

3° Let ε be as before and $\forall\, n \in \mathbb{N}\;\; x_n = \varepsilon$. Then $\forall\, n\;\; x_n \approx 0$, but $\lim\limits_{n\to\infty} x_n = \varepsilon \neq 0$. This fact does not contradict theorem 2, because the sequence $(x_n)_{n\in\mathbb{N}}$ is not standard.

4° For $n \approx \infty$ we have $n \cdot \sin 1/n \approx 1$, $(1 + 1/n)^n \approx e$, etc. Let $\mathcal{R}$ be the set of all convergent sequences $x \in \mathbb{R}^{\mathbb{N}}$. Such $\mathcal{R}$ is a partially ordered algebra on $\mathbb{R}$ relatively $+, \cdot, \le$, defined by

$$(\alpha x + \beta y)_n = \alpha x_n + \beta y_n, \quad (xy)_n = x_n y_n,$$
$$x \le y \Leftrightarrow (\exists\, n \in \mathbb{N})\ (\forall\, k > n)\ (x_k \le y_k),$$

here $x, y \in \mathcal{R},\ \alpha, \beta \in \mathbb{R}$.

The main result concerning limits is the following.

Theorem 3
The map lim *is a homomorphism of the partially ordered algebra $\mathcal{R}$ onto a linearly ordered field $\mathbb{R}$, i.e.*

$$\lim_{n\to\infty} (\alpha x_n + \beta y_n) = \alpha \lim_{n\to\infty} x_n + \beta \lim_{n\to\infty} y_n,$$
$$\lim_{n\to\infty} (x_n y_n) = \lim_{n\to\infty} x_n \cdot \lim_{n\to\infty} y_n,$$
$$x \le y \implies \lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n.$$

The non-standard *proof* of this theorem is almost trivial. At first, we note that theorem 3 is standard, therefore, by (T'), we only need to prove it for standard α, β, x, y. But for $x \in {}^{st}\mathbb{R}$ we have $\lim\limits_{n\to\infty} x_n = {}^\circ(x_\omega)$, $\omega \approx \infty$, and according to theorem 1.3° the map $r \mapsto {}^\circ r$ is a homomorphism of $\mathbb{F}$ onto ${}^{st}\mathbb{R}$. ▶

Now we will obtain something better than theorem 2.

Theorem 4

Let $x \in {}^{st}(\mathbb{R}^{\mathbb{N}})$. Suppose that for some $\omega \in \mathbb{N} \setminus {}^{st}\mathbb{N}$ and for all $n < \omega$ $\;n \approx \infty \implies x_n \approx 0$. Then x is convergent and $\lim\limits_{n\to\infty} x_n = 0$.

We need the following interesting lemma to prove the theorem.

Lemma 2

Let $k \in {}^{st}(\mathbb{N}^{\mathbb{N}})$ be strictly increasing and $\omega \in \mathbb{N}\setminus{}^{st}\mathbb{N}$. Then $(\exists\, n \in \mathbb{N}\setminus{}^{st}\mathbb{N})$ $(k_n < \omega)$. (This means that when n passes from values which are $\ll \infty$ to values which are $\approx \infty$, k cannot jump over $\omega \approx \infty$).

Proof. If $n \in {}^{st}\mathbb{N}$, then $k_n \in {}^{st}\mathbb{N}$ (the value of a standard function at any standard point is standard). Therefore, ${}^{st}\mathbb{N} \subseteq \tilde{N} := \{n \in \mathbb{N} :\; k_n < \omega\}$. Since the definition of $\tilde{N}$ is internal, the set $\tilde{N}$ is internal, too. By (9), there exists $n_\omega \in \tilde{N}$, $n_\omega \approx \infty$. As k is strictly increasing, $k_{n_\omega} \geq n_\omega$. Hence $k_{n_\omega} \approx \infty$ and $k_{n_\omega} < \omega$. ▶

Proof of theorem 4. Suppose that x is divergent. Then, by (T'), there exist a *standard* $\varepsilon > 0$ and a *standard*, strictly increasing $k \in \mathbb{N}^{\mathbb{N}}$, such that $\forall\, n \in \mathbb{N}\;\; |x_{k_n}| > \varepsilon$. By lemma 2, there exists $k_n < \omega$ which is infinite, contradicting the assumption $n \approx \infty \implies x_n \approx 0$. ▶

Corollary 5

Let $x \in {}^{st}(\mathbb{R}^{\mathbb{N}})$ and $\ell \in {}^{st}\mathbb{R}$. Suppose that for some $\omega \in \mathbb{N} \setminus {}^{st}\mathbb{N}$ and for all $n < \omega$ $\;n \approx \infty \implies x_n \approx \ell$. Then x is convergent and $\lim\limits_{n\to\infty} x_n = \ell$.

Indeed, the sequence $(x_n - \ell)_{n\in\mathbb{N}}$ satisfies the condition of example 14.

We recall that $\ell \in \mathbb{R}$ is said to be *a limit point* of $x \in \mathbb{R}^{\mathbb{N}}$ if

$$(\forall\, \varepsilon > 0)(\forall\, N \in \mathbb{N})(\exists\, n > N)(|x_n - \ell| < \varepsilon), \tag{13}$$

or, equivalently, if x contains a subsequence which converges to ℓ.

Theorem 5

Let $x \in {}^{st}(\mathbb{R}^{\mathbb{N}})$ and $\ell \in {}^{st}\mathbb{R}$. The number ℓ is a limit point of x if and only if $(\exists\, \omega \approx \infty)$ $(x_\omega \approx \ell)$.

Proof. Let ℓ be a limit point of x and let $k \in \mathbb{N}^{\mathbb{N}}$ be a strictly increasing sequence such that $\lim\limits_{n\to\infty} x_{k_n} = \ell$. By (T), we can choose a standard k. Then $(x_{k_n})_{n\in\mathbb{N}}$ is standard and, by (13), $\forall\, n \approx \infty\;\; x_{k_n} \approx \ell$.

Conversely, let $x_\omega \approx \ell$ for some $\omega \approx \infty$. Choose standard $\varepsilon > 0$ and $N \in \mathbb{N}$. Then, there exists $n > N$ such that $|x_n - \ell| < \varepsilon$ (namely, $n \approx \infty$). By (T'), we have (13). ▶

Exercise 9

1° Let $x \in {}^{st}(\mathbb{R}^{\mathbb{N}})$ be decreasing and positive. Assume that $x_{\omega_0} \approx 0$ for some $\omega_0 \approx \infty$. Prove that $\forall\, \omega \approx \infty \;\; x_\omega \approx 0$.

2° Construct $x \in {}^{st}(\mathbb{R}^{\mathbb{N}})$ which is divergent, but $x_n \approx 0$ for $\omega_1 < n < \omega_2$, where $\omega_1 \approx \infty$ and $\omega_2 - \omega_1 \approx \infty$.

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