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Factorial number system as an example of substantial learning environment for pre- and in-service teachers of mathematics*

Abstract. This article draws on the work of Wittmann and his followers who conceived and developed the notion of substantial learning environment (SLE). The paper contains a proposal of a teaching unit based on the definition of Factorial Number System (FNS). First, we illustrate the process of conversion from FNS to the Decimal Number System (DNS) and back. Secondly, we provide theorems on the divisibility rules for several numbers in FNS. The main aim of this paper is to present FNS as an example of a mathematically rich environment wherein pre-service teachers of mathematics may be actively engaged in the process of discovering subjectively new mathematics.

1. Substantial learning environments for teachers

The notion of substantial learning environment (SLE) introduced by Wittmann (1995) refers to educational environment which meets the following criteria:

1. It represents central objectives, contents and principles of teaching mathematics at a certain level.
2. It is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities.
3. It is flexible and can be adapted to the special conditions of a classroom.
4. It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research (Wittmann, 1995, p. 365-6).

SLEs by definition extend the boundaries of school level by linking curriculum contents with more advanced mathematics. Hence, they can and should also be

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explored by pre- and in-service teachers of mathematics, either within academic courses or along other ways of professional development. Although Wittmann (2001) emphasizes the important role that curriculum related topics play in student teachers' vocational preparation, he claims that above all, students should experience mathematics as an activity. It means that, also while attending courses on abstract mathematics, like for instance algebra or functional analysis, students should be given the opportunity to discover subjectively new mathematics in their own effective struggles. Oftentimes, however, students become familiar with higher mathematics following the 'definition – theorem – proof' scheme, which leaves almost no room for their own exploration and invention. It seems that there is an urgent need of creating learning environments that could be substantial, in the sense of the above description, not only for school students but also for the students of abstract mathematics. All the features formulated for SLEs crafted for school mathematics would still be valid at the academic level. In particular, the postulate of "going beyond the level" can be realized in the form of going beyond the artificially imposed boundaries of a given sub-discipline of mathematics. Experiencing pieces of knowledge not as loosely joint, but interrelated parts of mathematics, could make the learning of the subject more meaningful to the students.

We focus our attention on educational environments designed for the purpose of substantial learning, mathematical growth and professional development of pre- and in-service teachers of mathematics and we call them Substantial Learning Environments for Teachers (SLEfT).

Since the time when the notion of substantial learning environment has been formulated, many SLEs appropriate for the school level have been described and tested by different scholars (e.g., Wittmann, 2005; Krauthausen, Scherer, 2013; Nührenbörger et al., 2016; Kieran, 2018). In the following section of this article we share an example of a mathematically rich environment which, to our sense, may serve as an example of SLEfT. In section 6, we provide several arguments explaining why and how the Factorial Number System teaching unit meets the criteria of SLE

2. Factorial Number System – a teaching unit proposal

According to Wittmann (1984):

> Appropriate teaching units provide opportunities for doing mathematics, for studying one's own learning processes and those of students, for evaluating different forms of social organisation, and for planning, performing and analysing practical teaching. Therefore teaching units are a unique means for penetrating all components of teacher training and relating them to one another (p. 30).

Wittmann formulates a brief template for a teaching unit (TU) description which includes statements about TU objectives, materials, problems and background. Using this scheme to briefly describe our proposal, we outline the following elements:

Teaching Unit: Factorial Number System (FNS)

Objectives: converting numbers from FNS to decimal number system (DNS) and back, discovering theorems related to the divisibility rules in FNS

Materials: the definition of FNS and number notation in FNS

Problems: Given numbers in FNS, find their representation in DNS.
Given numbers in DNS, find their representation in FNS.
Exploring the number notation in FNS, find and formulate the divisibility rules in FNS.

Background: different representations of numbers, non-standard mixed base positional number system, discovering mathematical theorems.

Our choice of FNS is motivated by several arguments. First of all, it is a rich field wherein many mathematical investigations can be held (however, in this paper we narrow our focus only to the problems of conversion and discovering the divisibility rules in FNS). Secondly, the topic bridges school and academic mathematical education, since it evokes reflection on DNS, a system with which the students are familiar, a general notion of a number system which encompasses far more exemplifications, and also one of the fundamental mathematical and didactical ideas of representation.

3. Factorial Number System

Mixed base systems were a subject of interest to mathematicians, including George Cantor, already in the 19^{th} century. Generally speaking, they are non-standard positional numeral systems where the numerical base varies from position to position. For example, if we measured time of 2 years, 5 months, 3 weeks, 5 days, 8 hours and 35 minutes, we could code this time as a sequence: (2, 5, 3, 5, 8, 35), where each position is defined in terms of different base. The system known today as FNS was first introduced by Laisant (1888). The notion of "factorial number system" was used, for instance, by Knuth (1981).

Definition 1
Factorial Number System is a positional, mixed base number system in which the multipliers for numbers at the subsequent positions are factorials. We use the subscript "!" to denote a number written in FNS.

Let a be a natural number such that when written in FNS it takes the form of an n-digit number:

$$(a)_! = (a_n a_{n-1} ... a_1)_!. \tag{1}$$

The right side of this equation is an abbreviation of the following combination of factorials and natural coefficients:

$$(a)_! = (a_n a_{n-1} \ldots a_1)_! = a_n \cdot n! + a_{n-1} \cdot (n-1)! + \ldots + a_1 \cdot 1! \tag{2}$$

For each multiplier $i!$, $i \in \{1, 2, ..., n\}$ the coefficient $a_i \in \mathbb{N}$ satisfies the inequality: $0 \leq (a_i)_{10} \leq i$.

There exist two definitions of FNS which differ only by the first multiplier from the right side. It may be equal to 0! and then every natural number ends with 0, or it may be equal to 1!, as well. In our investigations we adopt the latter version.

A set of FNS digits is infinite but countable. A digit congruent with 10 is denoted as A. The subsequent digits, i.e. 11, 12 etc. are denoted as B, C, etc. respectively.

The biggest number recorded in FNS with the use of digits $\{0, 1, 2, ..., 9\}$ is $(987654321)_!$ which equals $(3628799)_{10}$. This equality may be shown from the definition, but it may be also justified by the following theorem:

Theorem 1
For all $n \in \mathbb{N}_1$, is true

$$\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1. \tag{3}$$

This theorem means that the biggest number that can be represented with n digits in FNS is equal to the smallest number that can be represented with $n+1$ digits minus 1.

Proof:
Let $n \in \mathbb{N}_1$, then

$$\begin{aligned}\sum_{i=1}^{n} i \cdot i! &= \sum_{i=1}^{n}((i+1) \cdot i! - i!) = \sum_{i=1}^{n}(i+1)! - \sum_{i=1}^{n} i! \\ &= (n+1)! + \sum_{i=1}^{n-1}(i+1)! - \sum_{i=1}^{n} i! \\ &= (n+1)! + \sum_{i=1}^{n-1}(i+1)! + 1! - \sum_{i=1}^{n} i! - 1! \\ &= (n+1)! + \sum_{i=1}^{n} i! - \sum_{i=1}^{n} i! - 1 = (n+1)! - 1,\end{aligned}$$

what was to be demonstrated.

We will apply this theorem to find the decimal value of $(987654321)_!$.

Example 1

$$\begin{aligned}(987654321)_! = 9 \cdot 9! + 8 \cdot 8! + ... + 2 \cdot 2! + 1 \cdot 1! = \sum_{i=1}^{9} i \cdot i! = (9+1)! - 1 \\ = 10! - 1 = 3628800 - 1 = 3628799.\end{aligned}$$

Let us notice here, an interesting resemblance. In DNS the biggest $n+1$-digit number that we can obtain using all the digits from 0 to 9, would be written with n nines as 99...999. The value of this number equals

$$9 \cdot 10^n + 9 \cdot 10^{n-1} + ... + 9 \cdot 10^2 + 9 \cdot 10^1 + 9 \cdot 10^0 = 10^{n+1} - 1.$$

4. Conversion from FNS to DNS and back

4.1. Conversion FNS for DNS

4.1.1. Method I

The easiest way to convert a number from FNS to DNS is to use Definition 1. Hence, in a general case we will obtain the decimal value of a given n-digit number as follows:

$$(a_n a_{n-1} \dots a_2 a_1)_! = a_n \cdot n! + a_{n-1} \cdot (n-1)! + \dots + a_2 \cdot 2! + a_1 \cdot 1!.$$

Let us check this on one example:

Example 2
Convert $(12120)_!$ *from DNS to FNS.*

$$\begin{aligned}(12120)_! &= 1 \cdot 5! + 2 \cdot 4! + 1 \cdot 3! + 2 \cdot 2! + 0 \cdot 1! \\ &= 1 \cdot 120 + 2 \cdot 24 + 1 \cdot 6 + 2 \cdot 2 + 0 \cdot 1 \\ &= 178\end{aligned}$$

However, there are also some other interesting methods one can use to obtain the same result. One of them refers to an algorithm usually applied to polynomials.

4.1.2. Method II

The second method is based on the Horner's algorithm (see: e.g., Horner, 1819; Cajori, 1911) oftentimes used to calculate the polynomial value[1].

For any $(a)_!$ expressed as stated in (2) the following equation holds:

$$\begin{aligned}&a_n \cdot n! + a_{n-1} \cdot (n-1)! + ... + a_1 = \\ &\qquad = (((a_n \cdot n + a_{n-1}) \cdot (n-1) + a_{n-2}) \cdot (n-2) + ...) \cdot 2 + a_1 \qquad (4)\end{aligned}$$

We can also use the recurrence formula:

$$\begin{aligned}s_1 &= a_n \cdot n + a_{n-1} \\ s_2 &= s_1 \cdot (n-1) + a_{n-2} \\ ... &= ... \\ s_n &= s_{n-1} + a_1.\end{aligned}$$

Then the number $(a)_!$ after the conversion will have the s_n form.

Let us illustrate this method with the conversion of the same number:

[1] The algorithm uses the least number of multiplications.

Example 3
Convert $(12120)_!$ *from FNS to DNS using the Horner's algorithm.*

$$\begin{aligned}(12120)_! &= (((1\cdot 5+2)\cdot 4+1)\cdot 3+2)\cdot 2+0\\ &= ((7\cdot 4+1)\cdot 3+2)\cdot 2\\ &= (29\cdot +2)\cdot 2\\ &= 89\cdot 2\\ &= 178.\end{aligned}$$

4.2. Conversion DNS for FNS

4.2.1. Method I

Converting a number a written in DNS to FNS we may start by asking what is the biggest $k \in \mathbb{N}$ such that:

$$k! \leq a \ \wedge \ (k+1)! > a.$$

When such a number is found, we divide a by $k!$ and subsequently, divide the obtained remainders by consecutive natural numbers smaller than k. We continue this process until we reach the final division by 1!.

The quotients obtained from the division by l, where $1 \leq l \leq k$ and $l \in \mathbb{N}$, give the coefficients to be multiplied by $l!$ in the factorial representation of the number.

The whole process is illustrated with the following example:

Example 4
Convert 178 *from DNS to FNS.*

$$\begin{aligned}178 : 5! &= \mathbf{1}, remainder : 58\\ 58 : 4! &= \mathbf{2}, remainder : 10\\ 10 : 3! &= \mathbf{1}, remainder : 4\\ 4 : 2! &= \mathbf{2}, remainder : 0\\ 0 : 1! &= \mathbf{0}, remainder : 0\end{aligned}$$

Finally, rewriting the bold numbers in the top–down order we obtain the factorial notation for $(178)_{10}$:

$$(178)_{10} = (12120)_!.$$

4.2.2. Method II

The second method is similar to Euclid's algorithm. If we choose to include 0! the first step is the following: $178 : 1 = 178$, *remainder* : 0. If we exclude 0 position from our considerations then, we simply skip this step, and follow the algorithm below:

Example 5
Convert 178 *from DNS to FNS.*

$$\begin{aligned} 178 : 2 &= 89, remainder : \mathbf{0} \\ 89 : 3 &= 29, remainder : \mathbf{2} \\ 29 : 4 &= 7, remainder : \mathbf{1} \\ 7 : 5 &= 1, remainder : \mathbf{2} \\ 1 : 6 &= 0, remainder : \mathbf{1} \end{aligned}$$

Writing down the remainders of partial divisions in the reverse order we obtain the result of the conversion from DNS to FNS. In this case, the number $(178)_{10}$ turns out to be $(12120)_!$ as in the previous example.

5. Divisibility rules in FNS

In the article (Górowski, Łomnicki, 2006) the authors provide several theorems on the divisibility rules in DNS which the students are able to uncover, unveil and prove. We propose going a step further and stimulate students' activity with the search of divisibility rules in FNS. It is easy to notice that:

$$\forall l \in \mathbb{N}_1 \; l | (a_n \ldots a_l \underbrace{00\ldots0}_{l-1-times})_!, \text{ where } n \in \mathbb{N}_l.$$

The above formula is based on the observation that number l appears explicitly for the first time in $l!$ as one of the multipliers. Moreover, we may notice that the test of divisibility by any positive natural number l requires the examination of the expression obtained after subtraction of the sum $a_n \cdot n! + a_{n-1} \cdot (n-1)! + \ldots + a_l \cdot l!$, where $n, l \in \mathbb{N}_1$ and $l \leq n$. Later on in our considerations we use the following notation:

$$\mathcal{Q}(l) := a_{l-1} \cdot (l-1)! + \ldots + a_2 \cdot 2! + a_1 \cdot 1!.$$

Generally we notice, that $l|(a)_!$ if and only if $l|\mathcal{Q}(l)$, where $l \in \mathbb{N}_1$.

Below we provide some divisibility rules in FNS together with their proofs. The first method we use is based on examining the $\mathcal{Q}(l)$ segments, and the second relies on congruence. Moreover, we precede the first two theorems with examples of operative proofs (Wittmann, 2009), i.e., pre-formal formulations of divisibility rules, inferred from the observation of the 'behavior' of concrete numbers and the generality of the observed patterns.

We may start exploring the divisibility rules in FNS with what we already know from DNS. For example: we know how to recognize a number divisible by 2 in DNS – its last digit has to be 0, 2, 4, 6 or 8. We may now convert some consecutive natural numbers from DNS to FNS and try to search for some repeating features of even numbers.

Table 1. Observable patterns of the first twelve natural numbers

Odd numbers		**Even numbers**	
DNS notation $(a)_{10}$	**FNS notation** $(b)_!$	**DNS notation** $(a)_{10}$	**FNS notation** $(b)_!$
a	b	a	b
1	1	2	10
3	11	4	20
5	21	6	100
7	101	8	110
9	111	10	120
11	121	12	200

Now, what we may observe is that all the numbers we converted end with 0 or 1 and that in the case of even numbers, the last digit is 0. We may also notice that since the last digit of a number in FNS representation is always either 0 or 1 (i.e., no other digit can stand on the last position), our conclusion has to be true also for any natural number different from the scrutinized examples. Hence we formulate the next theorem and provide a more formal proof for it.

Theorem 2 (divisibiltity by 2)
$2|(a)_! \Leftrightarrow a_1 = 0.$

Proof:
Let us notice that

$$(a)_! = \underbrace{a_n \cdot n! + a_{n-1} \cdot (n-1)! + \ldots + a_3 \cdot 3! + a_2 \cdot 2!}_{\textit{this sum is divisible by } 2,\ \textit{because every multiplier contains } 2} + \underbrace{a_1 \cdot 1!}_{Q(2)}.$$

Number $(a)_!$ is divisible by 2 if and only if $2|Q(2)$. Since $a_1 \in \{0, 1\}$, then a_1 must equal 0.

Let us begin our attempts toward formulation of the divisibility rule for 3 in FNS, from taking a closer look at the FNS representations of several natural numbers divisible by 3.

Table 2. Observable patterns of the first ten natural numbers divisible by 3

Natural number divisible by 3	
DNS notation $(a)_{10}$	**FNS notation** $(b)_!$
a	b
3	11
6	100
9	111
12	200
15	211
18	300
21	311
24	1000
27	1011
30	1100

The above examples lead us to a conclusion that among the numbers which in DNS do not exceed 30, all the FNS representations of numbers divisible by 3 end with digits 00 or 11. We may express this condition in other words and say that all these numbers, when represented in FNS, have the two last digits equal, that is $a_2 = a_1$. But how can we make sure that this pattern holds true in a general case? In order to check the validity of this regularity, let us do some simple arithmetic on the numbers represented in FNS. In DNS when moving from a natural number divisible by 3 to the next natural number sharing this property, we simply add 3. In FNS notation, number $(3)_{10}$ is represented as $(11)_!$. We already know that among the natural numbers from 1 to 30 (DNS), all the numbers divisible by 3 end with 00 or 11 (FNS). Let us start from an observation that in FNS when adding 3 to any of the numbers from Table 2 we have to do one of the additions below (Fig. 1):

$$\begin{array}{r} 0\,0 \\ +\ 1\,1 \\ \hline 1\,1 \end{array}$$

Fig. 1a.

$$\begin{array}{r} {\scriptstyle 1} \\ 1\,1 \\ +\ 1\,1 \\ \hline 1\,0\,0 \end{array}$$

Fig. 1b.

What holds true for all the considered examples, may now be smoothly extended to a general case derived from the numbers in the table. When adding 3 to 30 we receive a result shown in Fig 1a – number 33 in FNS is represented as $(1111)_!$. Now, when adding 3 to 33 we proceed according to the arithmetic shown in Fig 1b and the result is 36, represented as $(1200)_!$. We may now easily extrapolate this example forward and say that this pattern has to be repeated cyclically. This reasoning is a pre-formal proof of a theorem stating that if a number is divisible by 3, its representation in FNS has the two last digits equal (00 or 11). We skip the operative proof of the reverse implication, leaving it to the readers, and move on to a formal proof of the equivalence of the abovementioned conditions.

Theorem 3 (divisibiltity by 3)
$3|(a)_! \Leftrightarrow a_2 = a_1.$

The first proof of theorem 3:
Let us examine the number:

$$(a)_! = \underbrace{a_n \cdot n! + a_{n-1} \cdot (n-1)! + \ldots + a_4 \cdot 4! + a_3 \cdot 3!}_{\textit{this sum is divisible by } 3\textit{, because every multiplier contains } 3} + \underbrace{a_2 \cdot 2! + a_1 \cdot 1!}_{\mathcal{Q}(3)}.$$

The divisibility of number $(a)_!$ by 3 depends on the divisibility of number $a_2 \cdot 2! + a_1$ by 3:

$$3|(a)_! \Leftrightarrow 3|a_2 \cdot 2! + a_1 \cdot 1! \Leftrightarrow 3|2 \cdot a_2 + a_1$$

It is known, that $a_1 \in \{0, 1\}$. Since $a_2 \in \{0, 1, 2\}$, then $2 \cdot a_2 \in \{0, 2, 4\}$.

All the possible values of $2 \cdot a_2 + a_1$ are presented in Table 3:

Table 3. Possible values of $2 \cdot a_2 + a_1$.

a_2	$2 \cdot a_2$	a_1	$2 \cdot a_2 + a_1$	a_2	$2 \cdot a_2$	a_1	$2 \cdot a_2 + a_1$
0	0	**0**	0	0	0	1	1
1	2	0	2	1	2	1	**3**
2	4	0	4	2	4	1	5

We see that the sum $2 \cdot a_2 + a_1$ is divisible by 3 only in two cases: when $a_2 = a_1 = 0$ and when $a_2 = a_1 = 1$. Thus

$$3|(a)_! \Leftrightarrow (a_2 = a_1 = 0 \vee a_2 = a_1 = 1) \Leftrightarrow a_2 = a_1.$$

The last equivalence follows from the fact that a_1 takes only the values 0 or 1, what was to be demonstrated.

The second proof of theorem 3:
Let us examine the number $(a)_!$ in the form (4):

$$3|(a)_! \Leftrightarrow (((((a_n \cdot n + a_{n-1}) \cdot (n-1) + a_{n-2}) \cdot \\ \cdot (n-2) + \ldots) \cdot 4 + a_3) \cdot 3 + a_2) \cdot 2 + a_1 \equiv 0 \pmod 3$$

It is known, that

$$((((a_n \cdot n + a_{n-1}) \cdot (n-1) + a_{n-2}) \cdot (n-2) + \ldots) \cdot 4 + a_3) \cdot 3 \equiv 0 \pmod 3,$$

Thus using the properties of congruence we receive:

$$\begin{aligned} \mathcal{Q}(3) = 2 \cdot a_2 + a_1 \equiv 0 \pmod 3 &\Leftrightarrow 3 \cdot a_2 - a_2 + a_1 \equiv 0 \pmod 3 \\ &\Leftrightarrow -a_2 + a_1 \equiv 0 \pmod 3 \\ &\Leftrightarrow a_1 \equiv a_2 \pmod 3 \end{aligned}$$

Let us notice, that $|a_1 - a_2| \in \{0, 1, 2\}$, thus $a_1 - a_2 = 0$, since it is the only value in this set which is divisible by 3. This eventually brings the condition $a_2 = a_1$.

Using the theorems 2 and 3 we determine the divisibility rule for 6.

Theorem 4 (divisibiltity by 6)
$6|(a)_! \Leftrightarrow a_2 = a_1 = 0$.

Proof:
We know that $6|(a)_!$, if and only if $2|(a)_!$ and $3|(a)_!$. From the theorems 2 and 3 we infer that

$$6|(a)_! \Leftrightarrow (a_1 = 0 \wedge a_1 = a_2) \Leftrightarrow a_2 = a_1 = 0$$

what was to be demonstrated.

Theorem 5 (divisibiltity by 4)
$4|(a)_! \Leftrightarrow (2|a_3 + a_2 \wedge a_1 = 0)$.

Proof:
We have

$$(a)_! = \underbrace{a_n \cdot n! + a_{n-1} \cdot (n-1)! + \ldots + a_5 \cdot 5! + a_4 \cdot 4!}_{\textit{this sum is divisible by } 4,\ \textit{because every multiplier contains } 4} + \underbrace{a_3 \cdot 3! + a_2 \cdot 2! + a_1 \cdot 1!}_{\mathcal{Q}(4)}.$$

Then

$$4|(a)_! \Leftrightarrow 4|\mathcal{Q}(4) \Leftrightarrow 4|6 \cdot a_3 + 2 \cdot a_2 + a_1.$$

It is known, that natural number divisible by 4 has to be divisible by 2, then $a_1 = 0$. Thus

$$4|6 \cdot a_3 + 2 \cdot a_2 \Leftrightarrow 4|4 \cdot a_3 + 2 \cdot a_3 + 2 \cdot a_2 \Leftrightarrow 4|2 \cdot (a_3 + a_2) \Leftrightarrow 2|a_3 + a_2.$$

Eventually we receive:

$$4|(a)_! \Leftrightarrow (2|a_3 + a_2 \wedge a_1 = 0).$$

Theorem 6 (divisibiltity by 8)
$8|(a)_! \Leftrightarrow (a_3 = a_2 \wedge a_1 = 0)$.

Proof:
For number $(a)_!$ to be divisible by 8, a must be an even number, thus $a_1 = 0$. Let us examine the number

$$(a)_! = \underbrace{a_n \cdot n! + a_{n-1} \cdot (n-1)! + \ldots + a_5 \cdot 5! + a_4 \cdot 4!}_{\textit{this sum is divisible by } 8,\ \textit{because every multiplier contains } 8} + \underbrace{a_3 \cdot 3! + a_2 \cdot 2! + 0 \cdot 1!}_{\mathcal{Q}(4)}.$$

As a consequence:

$$8|(a)_! \Leftrightarrow 8|a_3 \cdot 3! + a_2 \cdot 2! \Leftrightarrow 8|6 \cdot a_3 + 2 \cdot a_2.$$

Let us also notice that we received similar result previously, when examining divisibility by 4. It is not surprising, since the numbers divisible by 8 can only be found among the numbers divisible by 4.

Since

$$6 \cdot a_3 + 2 \cdot a_2 = 2 \cdot (3 \cdot a_3 + a_1)$$

for number $(a)_!$ to be divisible by 8, it suffices that $4|3 \cdot a_3 + a_2$. All the possible values at $3 \cdot a_3 + a_2$ are presented in Table 4:

Table 4. Possible values of $3 \cdot a_3 + a_2$.

a_3	a_2	$3 \cdot a_3 + a_2$	a_3	a_2	$3 \cdot a_3 + a_2$	a_3	a_2	$3 \cdot a_3 + a_2$
0	0	**0**	0	1	1	0	2	2
1	0	3	1	1	**4**	1	2	5
2	0	6	2	1	7	2	2	**8**
3	0	9	3	1	10	3	2	11

Now we choose sums divisible by 4, that is: 0, 4 and 8.

Thus, the number $(a)_! = (a_n a_{n-1} ... a_1)_!$ is divisible by 8 if and only if:

$$a_1 = 0 \wedge a_3 = a_2.$$

Theorem 7 (divisibiltity by 5)
$5|(a)_! \Leftrightarrow 5|4 \cdot a_4 + a_3 + 2 \cdot a_2 + a_1$.

Proof:
Let us examine number $(a)_!$ in the form (4) again. Knowing that

$$((((a_n \cdot n + a_{n-1}) \cdot (n-1) + a_{n-2}) \cdot (n-2) + ...) \cdot 6 + a_5) \cdot 5 \equiv 0 \,(mod\, 5),$$

we receive condition $\mathcal{Q}(5) \equiv 0 \,(mod\, 5)$. Thus:

$$\begin{aligned} ((a_4 \cdot 4 + a_3) \cdot 3 + a_2) \cdot 2 + a_1 &\equiv 0 && (mod\, 5) \\ 24 \cdot a_4 + 6 \cdot a_3 + 2 \cdot a_2 + a_1 &\equiv 0 && (mod\, 5) \\ 20 \cdot a_4 + 4 \cdot a_4 + 5 \cdot a_3 + a_3 + 2 \cdot a_2 + a_1 &\equiv 0 && (mod\, 5) \\ 4 \cdot a_4 + a_3 + 2 \cdot a_2 + a_1 &\equiv 0 && (mod\, 5) \end{aligned}$$

Using the definition of congruence, we finally have:

$$5|4 \cdot a_4 + a_3 + 2 \cdot a_2 + a_1.$$

Following the steps analogically, we can specify the divisibility rules for 7, 9 and 10, leaving the proofs to the reader.

Theorem 8 (divisibiltity by 7)
$7|(a)_! \Leftrightarrow 7|6 \cdot a_6 + a_5 + 3 \cdot a_4 + 6 \cdot a_3 + 2 \cdot a_2 + a_1$.

Theorem 9 (divisibiltity by 9)
$9|(a)_! \Leftrightarrow (3|a_5 + 2 \cdot a_4 + 2 \cdot a_3 + a_2 \wedge a_2 = a_1)$.

Theorem 10 (divisibiltity by 10)
$10|(a)_! \Leftrightarrow (10|4 \cdot a_4 + 6 \cdot a_3 + 2 \cdot a_2 \wedge a_1 = 0) \Leftrightarrow (5|2 \cdot a_4 + 3 \cdot a_3 + a_2 \wedge a_1 = 0)$.

Some of the tests of divisibility, as being practiced in school, relate to the occurrence of particular digits on specified positions of numbers, for example: *The natural number is divisible by* 5 *if and only if its last digit is* 0 *or* 5. Comparing this approach to the test of divisibility in FNS, we state that similar conditions are still possible, but far less effective. Let us consider for example divisibility by 5. We know that in order to test whether a number is divisible by 5 it is sufficient to check if the value of $\mathcal{Q}(5)$ expression is divisible by 5. However, the value of this expression varies from 0 to 119. The set of natural numbers $\{0, 1, 2, ..., 119\}$ contains 24 numbers divisible by 5. Hence, if we wanted to determine all the configurations of digits a_4, a_3, a_2 and a_1 resulting in a number divisible by 5, we would obtain 24 different sequences. In case of any given number, it is not reasonable to check whether its 4 last digits match one of the 24 cases – a far more reasonable method would be to simply use Theorem 7.

For the readers' convenience we gather all the theorems in the table below:

Table 5. Selected divisibility rules in FNS

A number $(a_n a_{n-1} ... a_1)_!$ is divisible by	2	if and only if	$a_1 = 0$
	3		$a_2 = a_1$
	4		$2 \mid a_3 + a_2 \wedge a_1 = 0$
	5		$5 \mid 4 \cdot a_4 + a_3 + 2 \cdot a_2 + a_1$
	6		$a_2 = a_1 = 0$
	7		$7 \mid 6 \cdot a_6 + a_5 + 3 \cdot a_4 + 6 \cdot a_3 + 2 \cdot a_2 + a_1$
	8		$a_3 = a_2 \wedge a_1 = 0$
	9		$3 \mid a_5 + 2 \cdot a_4 + 2 \cdot a_3 + a_2 \wedge a_2 = a_1$
	10		$5 \mid 2 \cdot a_4 + 3 \cdot a_3 + a_2 \wedge a_1 = 0$

It is often said that hardly ever are mathematical ideas presented in such a way that one can track the zig-zag path followed through by the author. We are fully aware of the fact that this statement holds true also in the case of our work. Our aim, however, was to briefly present the substantial, mathematical content to be found along the intellectual play with FNS. We hope that the interested readers will try to discover the above theorems in their own way. And by doing so, i.e. by exploring the factorial number system, they will surely discover the plethora of mathematical activities standing behind the presented work.

6. Factorial Number System as a substantial learning environment for pre-service teachers

In this section, we provide several arguments supporting our assertion that Factorial Number System teaching unit meets the criteria of SLE.

6.1. FNS represents central objectives, contents and principles of teaching school mathematics and that of vocational preparation of pre-service teachers of mathematics

In the case of the proposed teaching unit, the starting point is the definition of FNS, a mixed base, positional number system, which rather does not appear in the traditional curriculum neither at high school, nor at the academic level. It is, however, connected with the topics covered by the school curriculum (decimal number system, which is the basic, but one of the many possible, notational systems one may consider, serving the purpose of representing the same numbers in different ways) as well as included in academic syllabi (i.e., non-decimal number systems). The definition of FNS lays the ground for a particular mathematical micro-world (or a problem field, see: Pehkonen, 1992; Solvang, 1994) to be explored by the pre-service teachers. The first activity the student teachers take is the conversion of numbers written in FNS into the DNS notation and back. This initial activity is necessary for students to learn "how the system works" and it reveals the basic properties of the system. Then students explore the emerging structure of FNS in order to find out the divisibility rules for several numbers in this system. Students formulate and test hypotheses and discover theorems. The best way to proceed with this task is to work in groups wherein partial results are communicated and negotiated. The last phase includes communicating the results to the members of other teams and an open discussion where the collected data may be compared. Such activities encapsulate the general objectives of mathematical education mentioned in (Winter, 1975), i.e. mathematizing, exploring, reasoning and communicating.

At our university, student teachers meet non-decimal, fixed base number systems on three kinds of courses: *didactics of mathematics* – where they explore standard, but other than base ten number systems in order to better understand both the nature of DNS and school students' difficulties with it, *elementary numerical methods* – where one of the basic topics refers to the conversion between and arithmetic operations within different standard number systems, and *informatics* – where the topics from mathematical courses are extended and shaped for the purposes of IT teaching (only for those students who choose a specialization "mathematics with informatics"). Assuming that at other universities prospective teachers of mathematics attend courses that cover more or less similar topics, we believe that hardly ever are students introduced to FNS. Exploring FNS, students may experience how much they rely on intuitions derived from the base ten system. But the new environment requires leaving some of the previous habits of mind, and finding out the way the new system works. Due to its non-standard nature, FNS brings in a lot of newness and freshness which in turn may evoke students' puzzlement and curiosity. The fact that the students receive only the definition of the system and they are given only the basic number notation, prevents learners from being passive recipients of ready-made knowledge. They rather need to become knowledge builders instead. This may be an important factor supporting an inquisitive learning (i.e., intrinsically motivated). All in all, FNS creates a ground for learning which starts with interest and wondering, leads through researching a new field, inquiring and reasoning, towards discovery based on collected findings and noticing new emerging questions to be investigated (Scardamalia, 2002).

Such a way of introducing student teachers to a new topic fosters building a bridge between theoretical knowledge and practice. As stated by Wittmann (1998):

> Mathematical concepts are neither innate nor readily acquired through experience and teaching. Instead the learners have to reconstruct them in a continued social process where primitive and only partly effective cognitive structures which are chequered with misconceptions and errors gradually develop into more differentiated, articulated and coordinated structures which are better and better adapted to solving problems (pp. 149–150).

When investigating the divisibility rules in FNS, pre-service teachers may experience that the process of mathematical concepts acquisition requires time within which the concepts develop. This very practical, lived experience may be an important factor in students' professional development. Whenever student teachers study school curriculum related topics, they inevitably look at them through the lens of their contemporary knowledge and understanding of these topics. Even if they have some memories of the difficulties they have experienced as school students, it is almost impossible for them to go back to these distant memories and uncover all the layers of cognitive and affective difficulties they have passed through. Pre-service teachers then need some new experiences, built on an easily accessible – neither to abstract, nor too detached from school settings – material, where they can experience learning difficulties anew, observe and scrutinize the obstacles that emerge within that process and conduct a metacognitive reflection on their own processes of learning.

As mentioned before, the frequent occurrence of a "definition-theorem-proof" scheme at the academic level leads to imitative learning, memorizing rules and learning procedures by heart, without questioning and with almost no room for asking why some procedures work, and some others do not. Lithner (2008, 2017) distinguished two types of reasoning that occur in the learning of mathematics: Algorithmic Reasoning (AR) and Creative Mathematically founded Reasoning (CMR). The former is characterized as following the previously provided methods of solution, where the students' task is to repeat some steps given in advance. AR is a means to practice and master the procedures to be learned and it may be helpful in obtaining results quickly. However, if the aim is to learn something new, students need to create their own methods to solve problems, and no algorithms are given at hand. In such case, CMR is more relevant. This kind of reasoning meets the following criteria:

- Novelty / Creativity – the students create a reasoning sequence that they have never experienced before or re-create a forgotten one,

- Plausibility – there exist arguments supporting students' choice of strategy and justifications, also it is possible to judge the obtained results as true or false,
- Mathematical foundation / Anchoring – the arguments students use and the reasoning they conduct are anchored in intrinsic properties of the investigated object.

Whereas AR is supported by the imitative reasoning (i.e., repetition of already known procedures), CMR requires creative thinking and discovering subjectively new pathways. If the aim is to obtain results quickly, pre-existing algorithms do make the task easier. They also free some space in the working memory of the learners, so that they could focus on more demanding and more complex problems. AR serves well also in tasks oriented on practicing skills. If, however, the aim is to learn something new, develop mathematical competence, or acquire some new skills in problem solving and obtain deeper understanding of mathematical concepts, CMR is definitely far more effective. According to Lithner (2017): "It is the domination of algorithmic solution templates in mathematics teaching and learning, not the algorithms themselves, that is problematic" (p. 939). Some reasonable balance is needed here and it is the teachers' responsibility to carefully examine the context before they decide which way – AR or CMR – is more beneficial to their students in a particular situation (Jonsson et al., 2014, 2016).

As mentioned above, non-decimal standard, fixed base systems are oftentimes addressed at different courses for pre-service teachers of mathematics, thus it is likely that the students have already learned some algorithms enabling them operating within these systems (see: Wardrop, 1972; Fomin, 1974). It is very unlikely, however, that they have been introduced to FNS, hence they may begin their exploration from the very beginning, being equipped only with the definition of the system and the number notation. The nature of FNS itself then favours Creative Mathematically founded Reasoning. Paraphrasing famous words of Kant we could say that "Procedures without meaning are empty, meanings without procedures are blind". Whereas students may know many procedures, but lack their meaning, as well as having tasted some meanings still miss accurate procedures to arrange them, FNS creates an environment wherein meanings and procedures arise and develop hand in hand.

Another aspect of the proposed teaching unit is that when exploring FNS, students engage in a productive cognitive struggle (see: Granberg, 2016). It is well known that people make a better use of methods they discovered themselves, since the effects of learning by "finding out" last longer than the knowledge obtained by "being told" (Qwillbard, 2014; Wirebring et al., 2015; Norqvist, 2016). What inevitably becomes a part of the exploration of an unfamiliar territory is that the learners try making connections to their previous knowledge, for this is their only point of reference. In what follows, such activities support the development of relational understanding of some broader area the topic under investigation is a part of. These ideas correspond with the following words of Bruner (1977):

> The first object of any act of learning, over and beyond the pleasure it may give, is that it should serve us in the future... (A) way in which earlier learning renders later performance more efficient is through what is conveniently

> called nonspecific transfer or, more accurately, the transfer of principles and attitudes. In essence, it consists of learning initially not a skill but a general idea, which can be used as a basis for recognizing subsequent problems as special cases of the idea originally mastered (p. 17).

In the sense expressed by Bruner, we find a twofold role that FNS TU may play in the vocational preparation of mathematics teachers. One is that the experiences gained within the proposed activity may, through the nonspecific transfer addressed in the above excerpt, result in adopting a new perspective on the number systems one has met before. The other way, we believe such experiences may affect the learners' attitude, is that if the students find value in productive struggles with new problems and in exploring mathematical territories new to them, it is more likely that as in-service teachers they will be interested in offering activities of a similar nature to their own students.

6.2. FNS is related to significant mathematical contents, processes and procedures beyond school level, and as a rich source of mathematical activities can be adapted to the special conditions of a group of learners

Investigating properties of a new object, like factorial number system, students enter the world of intellectual work similar in its nature to the work of mathematicians – the scientists. At the beginning, the learners only have a piece of information, which turns out to give rise to genuine mathematical work. There arise some questions, but the students cannot know in advance how long it is going to take them before they will eventually find the answers. In fact, there is no guarantee they will find them. Also, in the case of solvable problems, it is hard to predict in advance the final solutions. There are various mathematical activities that become a part of the work of exploration: formulating and testing hypotheses, generalizing, formulating theorems and providing justifications for them, discovering algorithms and so forth.

From the mathematical perspective, FNS is rooted in number theory - a branch of higher mathematics devoted to the study of numbers, their properties and the relationships between them. The system provides an opportunity to consider and discuss one of the most fundamental ideas of mathematics, namely that of *representation* (Bruner, 1966; Lesh, Behr, Post, 1987a; Goldin, 2002). Representations make abstract mathematical concepts cognitively accessible to the learners. The development of mathematical competencies of the students entails obtaining fluency in interpreting and making effective use of representations of mathematical concepts. Metaphorically speaking, we could compare the different number systems – representing, in fact, the same numbers, just in different manners – to the different languages, enabling people communicating their thoughts. Knowing and understanding different representations of the same object, being able to recognize similarities and differences between them as well as knowing and understanding the process of conversion between them are one of the key elements of mathematical communication. Moreover, using different representations of the same object and knowing that they are not the object itself, enables taking different perspectives on the object (Tripathi, 2008). Meanwhile, many people identify numbers with their decimal representation. To them a number is not an abstract concept, only a sequence of digits. As a result, arithmetic operations become nothing more than just formal manipulations on symbols, they lose their meaning and sense. ”It turns out that the best way to deepen one's understanding of the decimal number system is to consider analogous problems for non-decimal systems” (Puchalska, Semadeni, 1988, p. 92). Also, learning about non-decimal systems enables learners to see how other systems are either different or similar to DNS. In particular, analyzing number representations in different systems with respect to their features, opens a room for discussing the transparency of number representations (Lesh, Behr, Post, 1987b; Zazkis, Gadowsky, 2001; Zazkis, Sirotic, 2010).

The divisibility by a certain number may be understood as a constant property of an abstract number, e.g. we may say that 18 is divisible by 9, always, regardless the form these two numbers take. In fact, what we deal with throughout the whole paper is the question of how to recognize and test divisibility of numbers in the context of a particular representation. Interestingly, when searching for divisibility rules in FNS, we strongly rely on our intuitions shaped and formed by the experiences we have with the divisibility rules in DNS. Thus, we have to admit that we derive knowledge about the behaviour of a particular representation of a natural number from the behaviour of another representation of this number, not from the abstract number itself. We may find a lot of joy and satisfaction playing with different representations of numbers, but do we get anyhow closer to the abstract concept of a number? Such questions give us a good lesson of humility and respect since they help us realize how limited we are when trying to reveal the secrets of the universe of abstract mathematical concepts.

6.3. FNS TU integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research

According to what has been said thus far, the proposed teaching unit may be used as a topic covered by one of the courses attended by pre-service teachers, depending on the purposes to be served. But there is one more way of making use of this topic. Within the time of vocational preparation, all the students are encouraged to become reflective practitioners (see: Wittmann, 2001). One of the ways to make this goal achievable leads through inculcating and developing some new habits of mind. A didactical reflection on one's own learning of abstract mathematical contents may lead to posing some questions, like for instance:

- How can I convince my students that it is important to distinguish examples from operative and formal proofs? Can I think of any other examples of mathematical tasks where my students may tend to treat examples as sufficient justifications? How will I address their false convictions?
- What new things have I learned about the topic I knew before (e.g., divisibility rules, number systems) from looking at it from a new perspective, that I have not used before?
- How does this topic refer to what we teach school students?

- How could a low threshold look like, if I wanted to offer my students a task whose high ceiling would correspond to this topic?
- What kind of obstacles have I encountered on my own way to understanding this topic / solving this problem?
- What can I learn from my own "getting stuck" moments? What have I found helpful and can I use it in my teaching in order to effectively help my future students to overcome the obstacles they will encounter?

Summary

Factorial number system is, as we believe, a good learning environment for the teachers to develop not only mathematical knowledge and skills, but also a kind of pedagogical mindfulness and sensibility.

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